Supplement to "The Term Structure of Macroeconomic Risks at the Effective Lower Bounds"

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A Proofs and Parameter estimates

A.1 The Gamma-zero (γ_0) distribution

The gamma-zero autoregressive process was introduced by Monfort et al. (2017) as a generalization of the autoregressive gamma process of Gouriéroux and Jasiak (2006). Let $\mathfrak{I}_t = \mathfrak{I}(X_t, z_{t-1})$ be a non-negative process which is a function of the risk factors X_t and z_{t-1}, and j_t be a Poisson variable with intensity $\mathfrak{I}_t$. z_t is conditionally gamma-zero distributed if:

$$j_t|\underline{X_t}, \underline{z_{t-1}} \sim \mathcal{P}\left(\mathfrak{I}(\underline{X_t}, \underline{z_{t-1}})\right) \quad \text{and} \quad z_t|j_t \sim \text{Gamma}_{j_t}(c) \ , \tag{A.1}$$

that is, conditionally on the Poisson mixing variable, z_t has a gamma distribution with shape (or degree of freedom) parameter j_t and a scale parameter c. When $j_t = 0$, the conditional distribution of z_t converges to a Dirac point mass at zero. Integrating with respect to j_t, we obtain the conditional distribution of z_t given X_t and z_{t-1} that is called gamma-zero, encompassing a zero point mass. The conditional distribution of z_t given X_t and its past can be expressed with its conditional Laplace transform:

$$\mathbb{E}\left[\exp(u_z z_t)\big|\underline{X_t}, \underline{z_{t-1}}\right] = \exp\left(\frac{u_z c}{1 - u_z c}\mathfrak{I}_t\right) \ , \tag{A.2}$$

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In this paper, we consider an intensity which is a linear-quadratic function of X_t and a linear function of z_{t-1} (see Equation (4)):

$$\mathbb{E}\left[\exp(u_z z_t)\big|\underline{X_t}, \underline{z_{t-1}}\right] = \exp\left(\frac{u_z c}{1-u_z c}\left(\alpha + \phi z_{t-1} + \kappa\beta' X_t + (\beta' X_t)^2\right)\right).$$

The properties of the gamma-zero are such that its first two conditional moments are linear in its underlying intensity $\mathfrak{I}_t$:

$$\mathbb{E}\left(z_t\big|\underline{X_t}, \underline{z_{t-1}}\right) = c\,\mathfrak{I}_t \quad \text{and} \quad \mathbb{V}\left(z_t\big|\underline{X_t}, \underline{z_{t-1}}\right) = 2\,c^2\,\mathfrak{I}_t\,.$$

We can expand the function of X_t in the intensity:

$$\begin{aligned}
\kappa\beta' X_t + (\beta' X_t)^2 &= \kappa\beta'\left(\mu + \Phi X_{t-1} + v_t\right) + \left[\beta'\left(\mu + \Phi X_{t-1} + v_t\right)\right]^2 \\
&= \kappa\beta'\left(\mu + \Phi X_{t-1} + v_t\right) + \left\{\left[\beta'\left(\mu + \Phi X_{t-1}\right)\right]^2 + \left(\beta' v_t\right)^2 + 2\beta'\left(\mu + \Phi X_{t-1}\right)\beta' v_t\right\} \\
&= \kappa\beta'\mu + \left(\beta'\mu\right)^2 + \beta'\Sigma\beta + \kappa\beta'\Phi X_{t-1} + \left(\beta'\Phi X_{t-1}\right)^2 + 2\left(\beta'\mu\right)\left(\beta'\Phi X_{t-1}\right) \\
&+ \quad \kappa\beta' v_t + \left[\left(\beta' v_t\right)^2 - \beta'\Sigma\beta\right] + 2\left(\mu + \Phi X_{t-1}\right)'\beta\beta' v_t
\end{aligned}$$

Using the conditional moments of linear-quadratic Gaussian processes (see technical Appendix B.2), the last row has zero conditional mean given the information available at $t-1$. The short-rate is then given by:

$$\begin{aligned}
r_t &= \underline{r} + c\,\mathfrak{I}_t + \varepsilon_t^z \\
&= \underline{r} + c\,\mathbb{E}_{t-1}\left(\mathfrak{I}_t\right) + \varepsilon_t^r \\
&= \underline{r} + c\left[\alpha + \phi z_{t-1} + \left(\kappa + \beta'\mu\right)\beta'\mu + \beta'\Sigma\beta + \left(\kappa + 2\mu'\beta\right)\beta'\Phi X_{t-1} + \left(\beta'\Phi X_{t-1}\right)^2\right] + \varepsilon_t^r \\
&= \underline{r} + \underbrace{c\left[\alpha + \left(\kappa + \beta'\mu\right)\beta'\mu + \beta'\Sigma\beta\right]}_{=\alpha^*} + \underbrace{c\phi}_{=\phi^*} z_{t-1} + c\left(\kappa + 2\mu'\beta\right)\beta'\Phi X_{t-1} + c\left(\beta'\Phi X_{t-1}\right)^2 + \varepsilon_t^r\,,
\end{aligned}$$

and by definition we obtain:

$$r_t = (1-\phi^*)\underline{r} + \alpha^* + \phi^* r_{t-1} + c\left(\kappa + 2\mu'\beta\right)\beta'\Phi X_{t-1} + c\left(\beta'\Phi X_{t-1}\right)^2 + \varepsilon_t^r\,, \qquad \text{(A.3)}$$

and

$$\varepsilon_t^r = c\left[\kappa\beta' v_t + \left[(\beta' v_t)^2 - \beta'\Sigma\beta\right] + 2\left(\mu + \Phi X_{t-1}\right)'\beta\beta' v_t\right] + \varepsilon_t^z\,. \qquad \text{(A.4)}$$

For conditional variance, we have:

$$\begin{aligned}
\mathbb{V}_{t-1}\left(r_t\right) &= c^2\mathbb{V}_{t-1}\left[\left(\kappa + 2\left(\mu + \Phi X_{t-1}\right)'\beta\right)\beta' v_t + \left((\beta' v_t)^2 - \beta'\Sigma\beta\right)\right] + \mathbb{V}_{t-1}\left(\varepsilon_t^z\right) \\
&= c^2\left(\left[\kappa + 2\left(\mu + \Phi X_{t-1}\right)'\beta\right]^2\beta'\Sigma\beta + 2\left(\beta'\Sigma\beta\right)^2\right) \\
&+ \quad 2c^2\left(\alpha + \phi z_{t-1} + (\kappa + \beta'\mu)\,\beta'\mu + \beta'\Sigma\beta + (\kappa + 2\mu'\beta)\,\beta'\Phi X_{t-1} + (\beta'\Phi X_{t-1})^2\right)\,,
\end{aligned}$$

which is a linear-quadratic function of X_{t-1}.

A.2 Affine $\mathbb{P}$-property

In this Section, we show that our physical dynamics are affine. Define $u = [u_x',\ \text{Vec}(U_x)',\ u_z']'$, where the blocks have respective size K, K^2 and 1. We first introduce the following Lemma.

Lemma A.1 *The conditional Laplace transform of* $[X_t', \text{Vec}(X_t X_t')']'$ *given its past is given by:*

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t\right)\left|\underline{X_{t-1}}\right.\right] \\
= \quad &\exp\left\{u_x'(I_K - 2\Sigma U_x)^{-1}\left(\mu + \frac{1}{2}\Sigma u_x\right) + \mu' U_x (I_K - 2\Sigma U_x)^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma U_x\right|\right. \\
+ \quad &\left.(u_x + 2U_x\mu)'(I_K - 2\Sigma U_x)^{-1}\Phi X_{t-1} + X_{t-1}\Phi' U_x (I_K - 2\Sigma U_x)^{-1}\Phi X_{t-1}\right\}
\end{aligned}$$

Proof See Cheng and Scaillet (2007). ■

Let us now calculate the conditional Laplace transform of $f_t := [X_t', \text{Vec}(X_t X_t')',\ z_t]$ given $\underline{f_{t-1}}$.

$$\mathbb{E}\left[\exp\left(u' f_t\right)\left|\underline{f_{t-1}}\right.\right] = \mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t + u_z z_t\right)\left|\underline{f_{t-1}}\right.\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left\{\mathbb{E}\left[\exp\left(u_x' X_t + X_t' U_x X_t + u_z z_t\right) | \underline{f_{t-1}}, X_t\right] | \underline{f_{t-1}}\right\} \\
&= \mathbb{E}\left[\exp\left\{u_x' X_t + X_t' U_x X_t + \frac{u_z c}{1-u_z c}\left[\alpha + \kappa\beta' X_t + X_t'\beta\beta' X_t + \phi z_{t-1}\right]\right\} | \underline{f_{t-1}}\right], \\
&= \exp\left(\frac{u_z c}{1-u_z c}(\alpha + \phi z_{t-1})\right)\mathbb{E}\left[\exp\left\{\left(u_x + \frac{\kappa u_z c}{1-u_z c}\beta\right)' X_t + X_t'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right) X_t\right\} | \underline{f_{t-1}}\right],
\end{aligned}
$$

We hence obtain the conditional Laplace transform of $[X_t', \text{Vec}(X_t X_t')']'$ applied in the two arguments $\left[\left(u_x + \kappa\frac{u_z c}{1-u_z c}\beta\right)'; \text{Vec}\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)'\right]'$. Using Lemma A.1, we have:

$$
\begin{aligned}
& \mathbb{E}\left[\exp\left(u' f_t\right) | \underline{f_{t-1}}\right] \\
= & \exp\left\{\frac{u_z c}{1-u_z c}(\alpha + \phi z_{t-1}) + \left(u_x + \frac{\kappa u_z c}{1-u_z c}\beta\right)'\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\left[\mu + \frac{1}{2}\Sigma\left(u_x + \frac{\kappa u_z c}{1-u_z c}\beta\right)\right]\right. \\
+ & \mu'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right| \\
+ & \left[\left(u_x + \frac{\kappa u_z c}{1-u_z c}\beta\right)' + 2\mu'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} \\
+ & \left. X_{t-1}'\Phi'\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\left[I_K - 2\Sigma\left(U_x + \frac{u_z c}{1-u_z c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1}\right\}. \qquad \text{(A.5)}
\end{aligned}
$$

This conditional Laplace transform is an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the physical measure. ■

A.3 Convexity adjustment for the pricing kernel

We now turn our interest to the derivation of the convexity adjustment in the pricing kernel given by Equation (7). By no-arbitrage, we have that:

$$
\mathbb{E}_{t-1}\left(M_t\right) = e^{-r_{t-1}} \Longleftrightarrow \mathbb{E}_{t-1}\left[\exp\left(\lambda_{t-1}' v_t + \lambda_r \varepsilon_t^r\right)\right] = \exp\left(\xi_{t-1}\right).
$$

To come back to the formulation of Equation (A.5), it is sufficient to multiply the left- and right-hand side of the previous Equation:

$$
\begin{aligned}
& \mathbb{E}_{t-1}\left[\exp\left(\lambda_{t-1}' v_t + \lambda_r \varepsilon_t^r\right)\right] = \exp\left(\xi_{t-1}\right) \\
\Longleftrightarrow \quad & \mathbb{E}_{t-1}\left[\exp\left(\lambda_{t-1}' X_t + \lambda_r z_t\right)\right] = \exp\left[\xi_{t-1} + \lambda_{t-1}'\left(\mu + \Phi X_{t-1}\right) + \lambda_r \mathbb{E}_{t-1}\left(z_t\right)\right] .
\end{aligned}
$$

Thus, using the result of Equation (A.5), we obtain:

$$
\begin{aligned}
\xi_{t-1} \;=\; & -\lambda_{t-1}'\left(\mu + \Phi X_{t-1}\right) - \lambda_r \mathbb{E}_{t-1}\left(z_t\right) + \frac{\lambda_r c}{1-\lambda_r c}(\alpha + \phi z_{t-1}) \\
+ \; & \left(\lambda_{t-1} + \frac{\kappa \lambda_r c}{1-\lambda_r c}\beta\right)' \left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1} \left[\mu + \frac{1}{2}\Sigma\left(\lambda_{t-1} + \frac{\kappa\lambda_r c}{1-\lambda_r c}\beta\right)\right] \\
+ \; & \mu'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\mu - \frac{1}{2}\log\left|I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right| \\
+ \; & \left[\left(\lambda_{t-1} + \frac{\kappa\lambda_r c}{1-\lambda_r c}\beta\right)' + 2\mu'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} \\
+ \; & X_{t-1}'\Phi'\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\left[I_K - 2\Sigma\left(\frac{\lambda_r c}{1-\lambda_r c}\beta\beta'\right)\right]^{-1}\Phi X_{t-1} .
\end{aligned}
$$

A.4 Risk-neutral affine property

To derive the risk-neutral conditional Laplace transform of f_t given $\underline{f_{t-1}}$, we use the transition formulas provided in Roussellet (2015), Chapter 4. Using the block recursive affine structure of f_t, the risk-neutral conditional Laplace transform of z_t given X_t and $\underline{f_{t-1}}$ is given by:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_z z_t\}\middle|X_t,\, \underline{f_{t-1}}\right) &= \frac{\mathbb{E}\left(\exp\left\{\left[u_z + \lambda_r\right] z_t\right\}\middle|X_t,\, \underline{f_{t-1}}\right)}{\mathbb{E}\left(\exp\left\{\lambda_r z_t\right\}\middle|X_t,\, \underline{f_{t-1}}\right)} \\
&= \exp\left\{\left(\frac{(u_z+\lambda_r)c}{1-(u_z+\lambda_r)c} - \frac{\lambda_r c}{1-\lambda_r c}\right)\left(\alpha + \kappa\beta' X_t + X_t'\beta\beta' X_t + \phi z_{t-1}\right)\right\} ,
\end{aligned} \tag{A.6}
$$

where $\mathbb{E}^{\mathbb{Q}}(\cdot)$ is the expectation operator under the risk-neutral measure. The difference of ratios can be simplified as follows.

$$\begin{aligned}\frac{(u_z+\lambda_r)c}{1-(u_z+\lambda_r)c}-\frac{\lambda_r c}{1-\lambda_r c} &= \frac{(1-\lambda_r c)\,(u_z+\lambda_r)c-[1-(u_z+\lambda_r)c]\,\lambda_r c}{[1-\lambda_r c]\,[1-(u_z+\lambda_r)c]}\\ &= c\frac{u_z-\lambda_r u_z c+u_z\lambda_r c}{[1-\lambda_r c]\,[1-(u_z+\lambda_r)c]}=\frac{u_z c}{[1-\lambda_r c]\,[1-(u_z+\lambda_r)c]}\,.\end{aligned}$$

Define now $c^{\mathbb{Q}}=\dfrac{c}{1-\lambda_r c}$, that is $c=\dfrac{c^{\mathbb{Q}}}{1+\lambda_r c^{\mathbb{Q}}}$. We obtain:

$$\frac{u_z c}{1-(u_z+\lambda_r)c} = \frac{u_z\frac{c^{\mathbb{Q}}}{1+\lambda_r c^{\mathbb{Q}}}}{1-(u_z+\lambda_r)\frac{c^{\mathbb{Q}}}{1+\lambda_r c^{\mathbb{Q}}}}=\frac{1+\lambda_r c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\times\frac{u_z c^{\mathbb{Q}}}{1+\lambda_r c^{\mathbb{Q}}}=\frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\,.$$

Hence the conditional Laplace transform of Equation (A.6) is given by:

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_z z_t\}\big|X_t,\,\underline{f_{t-1}}\right) &= \exp\left\{\frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\times\frac{\alpha+\kappa\beta' X_t+X_t'\beta\beta' X_t+\phi z_{t-1}}{1-\lambda_r c}\right\}\\ &=: \exp\left\{\frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\left(\alpha^{\mathbb{Q}}+\kappa^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}X_t+X_t'\beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}X_t+\phi^{\mathbb{Q}}z_{t-1}\right)\right\}\,.\end{aligned}$$

z_t is therefore conditionally gamma-zero distributed given X_t and its past, where the risk-neutral parameters are given by:

$$\alpha^{\mathbb{Q}} = \frac{\alpha}{1-\lambda_r c}\,,\quad \beta^{\mathbb{Q}}=\frac{\beta}{\sqrt{1-\lambda_r c}}\,,\quad \kappa^{\mathbb{Q}}=\frac{\kappa}{\sqrt{1-\lambda_r c}}\,,\quad \phi^{\mathbb{Q}}=\frac{\phi}{1-\lambda_r c}\,,\quad c^{\mathbb{Q}}=\frac{c}{1-\lambda_r c}$$

We turn now to the computation of the risk-neutral conditional Laplace transform of $(X_t',\,\mathrm{Vec}(X_tX_t')')'$ given $\underline{f_{t-1}}$. Again, using the property in Roussellet (2015) Chapter 4, we have:

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\{u_x'X_t+X_t'U_xX_t\}\,\big|\underline{f_{t-1}}\right) = \frac{\mathbb{E}\left[\exp\left\{(u_x+\widetilde{\lambda}_{t-1})'X_t+X_t'(U_x+\widetilde{\lambda}_r)X_t\right\}\,\big|\underline{f_{t-1}}\right)}{\mathbb{E}\left[\exp\left\{\widetilde{\lambda}_{t-1}'X_t+X_t'(U_x+\widetilde{\lambda}_r)X_t\right\}\,\big|\underline{f_{t-1}}\right)}\,,$$

where $\widetilde{\lambda}_{t-1}$ and $\widetilde{\lambda}_r$ are given by:

$$\widetilde{\lambda}_{t-1} \; = \; \lambda_0 + \beta \frac{\kappa \lambda_r c}{1-\lambda_r c} + \lambda_1 X_{t-1} \,, \quad \widetilde{\lambda}_r = \frac{\lambda_r c}{1-\lambda_r c} \beta\beta' \,.$$

The transition between the physical and risk-neutral dynamics of X_t are as if the SDF was exponential-quadratic, with adjusted prices of risk $\widetilde{\Lambda}_{t-1}$ and $\widetilde{\lambda}_r$. Since $\widetilde{\lambda}_r$ the price associated to $\mathrm{Vec}(X_t X_t')$ is constant through time, we can rely on the results of Monfort and Pegoraro (2012). We obtain that X_t follows a Gaussian VAR(1) under the risk-neutral measure and:

$$X_t = \mu^{\mathbb{Q}} + \Phi^{\mathbb{Q}} X_{t-1} + v_t^{\mathbb{Q}} \,,$$

where $v_t^{\mathbb{Q}} \overset{i.i.d.}{\sim} \mathcal{N}\left(0, \Sigma^{\mathbb{Q}}\right)$ is a Gaussian white noise, and $\mu^{\mathbb{Q}}$, $\Phi^{\mathbb{Q}}$ and $\Sigma^{\mathbb{Q}}$ are given by:

$$\begin{aligned}
\mu^{\mathbb{Q}} &= \left(I_K - 2\frac{\lambda_r c}{1-\lambda_r c} \Sigma\beta\beta' \right)^{-1} \left(\mu + \Sigma\lambda_0 + \frac{\kappa\lambda_r c}{1-\lambda_r c}\Sigma\beta \right) \\
\Phi^{\mathbb{Q}} &= \left(I_K - 2\frac{\lambda_r c}{1-\lambda_r c} \Sigma\beta\beta' \right)^{-1} \left(\Phi + \Sigma\lambda_1 \right) \\
\Sigma^{\mathbb{Q}} &= \left(I_K - 2\frac{\lambda_r c}{1-\lambda_r c} \Sigma\beta\beta' \right)^{-1} \Sigma \,.
\end{aligned}$$

The class of distributions are thus the same under the physical and the risk-neutral measure. Transforming Formula (A.5), the risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is given by:

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[\exp\left(u' f_t\right) | \underline{f_{t-1}} \right] \\
= \; & \exp\left\{ \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \left(\alpha^{\mathbb{Q}} + \phi^{\mathbb{Q}} z_{t-1} \right) - \frac{1}{2} \log\left| I_K - 2\Sigma^{\mathbb{Q}} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right| \right. \\
+ \; & \left(u_x + \frac{\kappa^{\mathbb{Q}} u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right)' \left[I_K - 2\Sigma^{\mathbb{Q}} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right]^{-1} \left[\mu^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}} \left(u_x + \frac{\kappa^{\mathbb{Q}} u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right) \right] \\
+ \; & \mu^{\mathbb{Q}'} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \left[I_K - 2\Sigma^{\mathbb{Q}} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right]^{-1} \mu^{\mathbb{Q}} \\
+ \; & \left[\left(u_x + \frac{\kappa^{\mathbb{Q}} u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right)' + 2\mu^{\mathbb{Q}'} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right] \left[I_K - 2\Sigma^{\mathbb{Q}} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right]^{-1} \Phi^{\mathbb{Q}} X_{t-1} \\
+ \; & \left. X_{t-1}' \Phi^{\mathbb{Q}'} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \left[I_K - 2\Sigma^{\mathbb{Q}} \left(U_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}} \beta^{\mathbb{Q}}\beta^{\mathbb{Q}'} \right) \right]^{-1} \Phi^{\mathbb{Q}} X_{t-1} \right\} . \qquad \text{(A.7)}
\end{aligned}$$

This conditional Laplace transform is an exponential-affine function of f_{t-1}. (f_t) is therefore an affine process under the risk-neutral measure. Combined with the fact that both r_t and π_t are affine functions of f_t augmented with the idiosyncratic inflation shocks ε_t^π, this is sufficient to define an affine term structure model (ATSM) (see e.g. Dai and Singleton (2000) or Darolles et al. (2006)).

A.5 Multi-horizon Laplace transform

Using the notation:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(u'f_t\right)\left|\underline{f_{t-1}}\right.\right] =: \exp\left\{\mathbb{A}^{\mathbb{Q}}(u) + \mathbb{B}^{\mathbb{Q}'}(u)X_{t-1} + X_{t-1}'\mathbb{C}^{\mathbb{Q}}(u)X_{t-1} + \mathbb{D}^{\mathbb{Q}}(u)z_{t-1}\right\},$$

where:

$$\begin{aligned}
\mathbb{A}^{\mathbb{Q}}(u) &= \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\alpha^{\mathbb{Q}} - \frac{1}{2}\log\left|I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right| \\
&+ \left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right)'\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\left[\mu^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}}\left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right)\right] \\
&+ \mu^{\mathbb{Q}'}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
\mathbb{B}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1'}\left[\left(u_x + \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\right) + 2\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)'\mu^{\mathbb{Q}}\right] \\
\mathbb{C}^{\mathbb{Q}}(u) &= \Phi^{\mathbb{Q}'}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(U_x + \frac{\kappa u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\beta^{\mathbb{Q}}\beta^{\mathbb{Q}'}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathbb{D}^{\mathbb{Q}}(u) &= \frac{u_z c^{\mathbb{Q}}}{1-u_z c^{\mathbb{Q}}}\phi^{\mathbb{Q}}
\end{aligned}$$

Since the one-period ahead conditional risk-neutral Laplace transform of f_t given $\underline{f_{t-1}}$ is exponential-affine in f_{t-1}, it is well-known that the conditional multi-horizon risk-neutral Laplace transform of $(f_t, \dots, f_{t+k})$ is also exponential-affine in f_{t-1} (see e.g. Darolles, Gourieroux, and Jasiak (2006)). We obtain:

$$\mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right)\left|\underline{f_{t-1}}\right.\right] = \exp\left(\mathbb{A}_k^{\mathbb{Q}}(u_0, \dots, u_k) + \mathbb{B}_k^{\mathbb{Q}'}(u_0, \dots, u_k)X_{t-1}\right.$$

$$+ \quad X'_{t-1}\mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k)\,X_{t-1} + \mathbb{D}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k)\,\mathfrak{r}_{t-1}\Big)\,,$$

where:

$$\begin{aligned}
\mathbb{A}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k) &:= \mathbb{A}^{\mathbb{Q}}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{B}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k) &:= \mathbb{B}^{\mathbb{Q}}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{C}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k) &:= \mathbb{C}^{\mathbb{Q}}_{k,k}(u_0,\ldots,\,u_k)\\
\mathbb{D}^{\mathbb{Q}}_k(u_0,\ldots,\,u_k) &:= \mathbb{D}^{\mathbb{Q}}_{k,k}(u_0,\ldots,\,u_k)\,,
\end{aligned}$$

with initial conditions $\mathbb{A}^{\mathbb{Q}}_{k,1}(u_0,\ldots,\,u_k) = \mathbb{A}^{\mathbb{Q}}(u_k)$, $\mathbb{B}^{\mathbb{Q}}_{k,1}(u_0,\ldots,\,u_k) = \mathbb{B}^{\mathbb{Q}}(u_k)$, $\mathbb{C}^{\mathbb{Q}}_{k,1}(u_0,\ldots,\,u_k) = \mathbb{C}^{\mathbb{Q}}(u_k)$ and $\mathbb{D}^{\mathbb{Q}}_{k,1}(u_0,\ldots,\,u_k) = \mathbb{D}^{\mathbb{Q}}(u_k)$, and $\forall i \in \{2,\ldots,k\}$,

$$\begin{aligned}
\mathbb{A}^{\mathbb{Q}}_{k,i}(u_0,\ldots,\,u_k) &= \mathbb{A}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\\
&+ \mathbb{A}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}\prime}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{B}^{\mathbb{Q}}_{k,i}(u_0,\ldots,\,u_k) &= \mathbb{B}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}\prime}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{C}^{\mathbb{Q}}_{k,i}(u_0,\ldots,\,u_k) &= \mathbb{C}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}\prime}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{D}^{\mathbb{Q}}_{k,i}(u_0,\ldots,\,u_k) &= \mathbb{D}^{\mathbb{Q}}\left(u_{k-i+1} + \left[\mathbb{B}^{\mathbb{Q}\prime}_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}^{\mathbb{Q}}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\,.
\end{aligned}$$

Since the conditional Laplace transform of f_t given $\underline{f_{t-1}}$ under the physical measure is the same function as the risk-neutral one, but plugging in the physical parameters instead of the risk-neutral ones, we easily obtain:

$$\begin{aligned}
&\varphi_{t-1}(u_0,\ldots,\,u_k) = \mathbb{E}\left[\exp\left(\sum_{i=0}^{k} u_i' f_{t+i}\right)|\underline{f_{t-1}}\right]\\
=:\quad &\exp\left(\mathbb{A}_k(u_0,\ldots,\,u_k) + \mathbb{B}_k'(u_0,\ldots,\,u_k)X_{t-1} + X'_{t-1}\mathbb{C}_k(u_0,\ldots,\,u_k)\,X_{t-1} + \mathbb{D}_k(u_0,\ldots,\,u_k)\mathfrak{r}_{t-1}\right)
\end{aligned} \tag{A.8}$$

where:

$$
\begin{aligned}
\mathbb{A}_k(u_0,\ldots,u_k) &:= \mathbb{A}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{B}_k(u_0,\ldots,u_k) &:= \mathbb{B}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{C}_k(u_0,\ldots,u_k) &:= \mathbb{C}_{k,k}(u_0,\ldots,u_k)\\
\mathbb{D}_k(u_0,\ldots,u_k) &:= \mathbb{D}_{k,k}(u_0,\ldots,u_k)\,,
\end{aligned}
$$

with initial conditions $\mathbb{A}_{k,1}(u_0,\ldots,u_k)=\mathbb{A}(u_k)$, $\mathbb{B}_{k,1}(u_0,\ldots,u_k)=\mathbb{B}(u_k)$, $\mathbb{C}_{k,1}(u_0,\ldots,u_k)=\mathbb{C}(u_k)$ and $\mathbb{D}_{k,1}(u_0,\ldots,u_k)=\mathbb{D}(u_k)$, and $\forall i\in\{2,\ldots,k\}$,

$$
\begin{aligned}
\mathbb{A}_{k,i}(u_0,\ldots,u_k) &= \mathbb{A}_{k,i-1}(u_0,\ldots,u_k)\\
&+ \mathbb{A}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{B}_{k,i}(u_0,\ldots,u_k) &= \mathbb{B}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{C}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\\
\mathbb{D}_{k,i}(u_0,\ldots,u_k) &= \mathbb{C}\left(u_{k-i+1}+\left[\mathbb{B}'_{k,i-1}(u_0,\ldots,u_k)\,,\mathrm{Vec}\left(\mathbb{C}_{k,i-1}(u_0,\ldots,u_k)\right)'\,,\mathbb{D}_{k,i-1}(u_0,\ldots,u_k)\right]'\right)\,.
\end{aligned}
$$

A.6 Pricing recursions

In this Section, we derive the pricing recursions for nominal bonds and TIPS. By no-arbitrage, we have:

$$
P_t^{(n)}=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-r_t}P_{t+1}^{(n-1)}\right]\quad\text{and}\quad P_t^{(n)^*}=\mathbb{E}_t^{\mathbb{Q}}\left[e^{-r_t}P_{t+1}^{(n-1)^*}\frac{\mathrm{CPI}_{t+1}}{\mathrm{CPI}_t}\right]
$$

We postulate the form given by Equation (14), that is:

$$
\begin{aligned}
P_t^{(n)} &= \exp\left(\mathcal{A}_n+\mathcal{B}_n'\,X_t+X_t'\,\mathcal{C}_n\,X_t+\mathcal{D}_n\,z_t\right),\\
P_t^{(n)^*} &= \exp\left(\mathcal{A}_n^*+\mathcal{B}_n^{*\prime}\,X_t+X_t'\,\mathcal{C}_n^*\,X_t+\mathcal{D}_n^*\,z_t\right)\,,
\end{aligned}
$$

Focusing first on nominal bonds, we obtain:

$$\mathcal{A}_n + \mathcal{B}_n' X_t + X_t' \mathcal{C}_n X_t + \mathcal{D}_n z_t \quad = \quad -\underline{r} - z_t + \mathcal{A}_{n-1} + \log \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(\mathcal{B}_{n-1}' X_{t+1} + X_{t+1}' \mathcal{C}_{n-1} X_{t+1} + \mathcal{D}_{n-1} z_{t+1} \right) \right]$$

Therefore, using the formulation of Equation (A.7), starting from initial conditions $\mathcal{A}_0 = 0$, $\mathcal{B}_0 = 0$, $\mathcal{C}_0 = 0$, $\mathcal{D}_0 = 0$, we get:

$$\begin{aligned}
\mathcal{A}_n &= -\underline{r} + \mathcal{A}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \alpha^{\mathbb{Q}} - \frac{1}{2} \log \left| I_K - 2\Sigma^{\mathbb{Q}} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \right| \\
&+ \left(\mathcal{B}_{n-1} + \frac{\kappa^{\mathbb{Q}} \mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right)' \left[I_K - 2\Sigma^{\mathbb{Q}} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \right]^{-1} \left[\mu^{\mathbb{Q}} + \frac{1}{2} \Sigma^{\mathbb{Q}} \left(\mathcal{B}_{n-1} + \frac{\kappa^{\mathbb{Q}} \mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right) \right] \\
&+ \mu^{\mathbb{Q}'} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \left[I_K - 2\Sigma^{\mathbb{Q}} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \right]^{-1} \mu^{\mathbb{Q}} \\
\mathcal{B}_n &= \Phi^{\mathbb{Q}'} \left[I_K - 2\Sigma^{\mathbb{Q}} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \right]^{-1'} \\
&\times \left[\left(\mathcal{B}_{n-1} + \frac{\kappa^{\mathbb{Q}} \mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \right) + 2 \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right)' \mu^{\mathbb{Q}} \right] \\
\mathcal{C}_n &= \Phi^{\mathbb{Q}'} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \left[I_K - 2\Sigma^{\mathbb{Q}} \left(\mathcal{C}_{n-1} + \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \beta^{\mathbb{Q}} \beta^{\mathbb{Q}'} \right) \right]^{-1} \Phi^{\mathbb{Q}} \\
\mathcal{D}_n &= \frac{\mathcal{D}_{n-1} c^{\mathbb{Q}}}{1 - \mathcal{D}_{n-1} c^{\mathbb{Q}}} \phi^{\mathbb{Q}} - 1 \,.
\end{aligned}$$

Now, turning to TIPS, there is a slight subtlety associated with the fact that π_{t+1} defines the year-on-year inflation and not the monthly inflation rate. The simplest case arises for maturities that are multiple of 12, (yearly maturities), which we have in the observables. In this case, the price of a TIPS with maturity $n = 12\widetilde{n}$ is given by:

$$P_t^{(12\widetilde{n})*} = \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(- \sum_{i=0}^{12\widetilde{n}-1} r_{t+i} \right) \frac{\mathrm{CPI}_{t+12\widetilde{n}}}{\mathrm{CPI}_t} \right] = \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(- \sum_{i=0}^{12\widetilde{n}-1} r_{t+i} \right) \exp \left(\sum_{j=1}^{\widetilde{n}} \pi_{t+12j} \right) \right] .$$

The pricing formulas are still closed-form but rely on the recursions given by the multi-horizon Laplace transform of the vector $f_t^{(aug)} = \left(X_t^{(aug)'}, X_t^{(aug)'} \otimes X_t^{(aug)'}, z_t \right)$ and $X_t^{(aug)} = \left(X_t', \varepsilon_t^{\pi}, \pi_{t-1}^* \right)'$. We obtain:

$$P_t^{(12\widetilde{n})*} = \exp \left(\widetilde{n} \bar{\pi} - 12 \widetilde{n} \underline{r} - z_t \right) \mathbb{E}_t^{\mathbb{Q}} \left[\exp \left(\sum_{i=1}^{12\widetilde{n}} u_i' f_{t+i}^{(aug)} \right) \right] , \tag{A.9}$$

where

$$u_{12\widetilde{n}} = \begin{pmatrix} \iota'_{K+2,K+2} & \iota'_{K+1,(K+2)^2} & 0 \end{pmatrix}' \tag{A.10}$$

$$u_i = \begin{pmatrix} \mathbf{0}'_{K+2} & \mathbf{0}'_{(K+2)^2} & -1 \end{pmatrix}' \quad \text{for} \quad i < 12\widetilde{n} \quad \text{and} \quad i/12 \neq \lfloor i/12 \rfloor \tag{A.11}$$

$$u_i = \begin{pmatrix} \iota'_{K+2,K+2} & \iota'_{K+1,(K+2)^2} & -1 \end{pmatrix}' \quad \text{for} \quad i < 12\widetilde{n} \quad \text{and} \quad i/12 = \lfloor i/12 \rfloor \tag{A.12}$$

where $\iota_{i,j}$ is the ith column of the identity matrix of size j. The recursions for the multi-horizon Laplace transform is detailed in Appendix A.5.

A.7 Liftoff probabilities

Using the properties of the gamma-zero process presented in Monfort et al. (2017), the probabilities for the short-term interest rate to stay at its lower bound for n periods are given by the following:

$$\begin{aligned} \mathbb{P}\left(r_{t+1:t+n} = \underline{r} \middle| \underline{f_t}\right) &= \mathbb{P}\left(z_{t+1:t+n} = 0 \middle| \underline{f_t}\right) = \lim_{v \to -\infty} \mathbb{E}\left[\exp\left(\sum_{i=1}^{n} v z_{t+i}\right) \middle| \underline{f_t}\right] \\ \mathbb{Q}\left(r_{t+1:t+n} = \underline{r} \middle| \underline{f_t}\right) &= \mathbb{Q}\left(z_{t+1:t+n} = 0 \middle| \underline{f_t}\right) = \lim_{v \to -\infty} \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\sum_{i=1}^{n} v z_{t+i}\right) \middle| \underline{f_t}\right] \end{aligned} \tag{A.13}$$

Notice that it is as if we were computing the n-maturity bond price and its respective expected component for a modified short-rate process. If our short-term nominal rate was given by $-vz_t$, this would be exactly the case. We thus only have to use the same pricing recursions as in Appendix A.6 but putting $\underline{r} = 0$ and replacing $c^{\mathbb{Q}}$ by $v \to +\infty$. We obtain:

$$\begin{aligned} \mathbb{Q}\left(r_{t+1:t+n} = \underline{r} \middle| \underline{f_t}\right) &= \exp\left(\mathcal{A}_n^{(\text{elb})} + \mathcal{B}_n^{(\text{elb})\prime} X_t + X_t' \mathcal{C}_n^{(\text{elb})} X_t + \mathcal{D}_n^{(\text{elb})} z_t\right) \\ \mathbb{P}\left(r_{t+1:t+n} = \underline{r} \middle| \underline{f_t}\right) &= \exp\left(\mathcal{A}_n^{\mathbb{P},(\text{elb})} + \mathcal{B}_n^{\mathbb{P},(\text{elb})\prime} X_t + X_t' \mathcal{C}_n^{\mathbb{P},(\text{elb})} X_t + \mathcal{D}_n^{\mathbb{P},(\text{elb})} z_t\right) . \end{aligned}$$

where the loadings follow the recursion:

$$\begin{aligned}
\mathcal{A}_n^{(\text{elb})} &= \mathcal{A}_{n-1}^{(\text{elb})} - \alpha^{\mathbb{Q}} - \frac{1}{2}\log\left|I_K - 2\Sigma^{\mathbb{Q}}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\right| \\
&+ \left(\mathcal{B}_{n-1}^{(\text{elb})} - \kappa^{\mathbb{Q}}\beta^{\mathbb{Q}}\right)'\left[I_K - 2\Sigma^{\mathbb{Q}}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\right]^{-1}\left[\mu^{\mathbb{Q}} + \frac{1}{2}\Sigma^{\mathbb{Q}}\left(\mathcal{B}_{n-1}^{(\text{elb})} - \kappa^{\mathbb{Q}}\beta^{\mathbb{Q}}\right)\right] \\
&+ \mu^{\mathbb{Q}\prime}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\right]^{-1}\mu^{\mathbb{Q}} \\
\mathcal{B}_n^{(\text{elb})} &= \Phi^{\mathbb{Q}\prime}\left[I_K - 2\Sigma^{\mathbb{Q}}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\right]^{-1\prime}\left[\left(\mathcal{B}_{n-1}^{(\text{elb})} - \kappa^{\mathbb{Q}}\beta^{\mathbb{Q}}\right) + 2\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)'\mu^{\mathbb{Q}}\right] \\
\mathcal{C}_n^{(\text{elb})} &= \Phi^{\mathbb{Q}\prime}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\left[I_K - 2\Sigma^{\mathbb{Q}}\left(\mathcal{C}_{n-1}^{(\text{elb})} - \beta^{\mathbb{Q}}\beta^{\mathbb{Q}\prime}\right)\right]^{-1}\Phi^{\mathbb{Q}} \\
\mathcal{D}_n^{(\text{elb})} &= -\phi^{\mathbb{Q}}.
\end{aligned}$$

For the physical probabilities, the recursions are the same replacing the risk-neutral parameters by the physical ones. We deduce the liftoff probabilities, i.e. to see the first interest rate hike in exactly $n+1$ periods.

$$\mathbb{P}\left(r_{t+1:t+n} = \underline{r}, r_{t+n+1} > \underline{r}\middle|\underline{f_t}\right) = \mathbb{P}\left(r_{t+1:t+n} = \underline{r}\middle|\underline{f_t}\right) - \mathbb{P}\left(r_{t+1:t+n+1} = \underline{r}\middle|\underline{f_t}\right).$$

A.8 Impulse Response Methodology

The affine structure of the model makes it easy to perform an impulse response analysis. All the variables considered in this section can be expressed as linear combinations of f_t components.[17] Let us consider the impact of a shock of size s of variable v_2 on variable v_1, where $v_1 = e'_{v_1}f_t$ and $v_2 = e'_{v_2}f_t$, with e_{v_1} and e_{v_2} vectors weighting and selecting the right entries of f_t depending on the variables of interest. Let us also denote by $\mathcal{E}_v = (e_{v_3}, \cdots, e_{v_q})$ the matrix of $(q-2)$ weighting vectors that define variables $v_j = e'_{v_j}f_t$ that we do not want to shock at the initial period. The impulse response at horizon n, denoted by $\mathcal{IRF}_{t,n}^{v_2\to v_1}$ is given by:

$$\begin{aligned}
\mathcal{IRF}_{t,n}^{v_2\to v_1} &= \mathbb{E}\left(e'_{v_1}f_{t+n}\middle|\underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right) \\
&- \mathbb{E}\left(e'_{v_1}f_{t+n}\middle|\underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right) \quad \text{(A.14)}
\end{aligned}$$

17. or equivalently, $f_t^{(aug)}$. we drop the superscript for ease of notation.

Using the semi-strong VAR formulation of Equation (IA.3), the impulse response function $\mathcal{IRF}_{t,n}^{v_2 \to v_1}$ is given by:

$$\begin{aligned}\mathcal{IRF}_{t,n}^{v_2 \to v_1} &= e'_{v_1}\Psi^n \Big[\mathbb{E}\left(f_t \big| \underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = s,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right) \\ &- \mathbb{E}\left(f_t \big| \underline{f_{t-1}},\, e'_{v_2}[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0,\, \mathcal{E}'_v[f_t - \mathbb{E}(f_t|\underline{f_{t-1}})] = 0\right)\Big] \end{aligned} \tag{A.15}$$

which only requires filtered values of the factor f_t given initial and observable conditions.

In our empirical exercise, we are in particular interested in shocking components of X_t itself for inflation central tendency or volatility shocks. The IRF of any variable v_1 to one of those structural shocks on $\iota'_i X_t$, ι_i selecting the i^{th} component of X_t is defined by:

$$\mathcal{IRF}_{t,n}^{v_2 \to v_1} = e'_{v_1}\Psi^n \begin{bmatrix} \Sigma^{1/2} & 0 \\ \Gamma_{t-1}\Sigma^{1/2} & \Sigma^{1/2} \otimes \Sigma^{1/2} \\ c\kappa\beta' & c(\beta \otimes \beta)' \end{bmatrix} \begin{bmatrix} s\iota_i \\ s^2\left(\iota_i \otimes \iota_i\right) \end{bmatrix}, \tag{A.16}$$

where Γ_{t-1} is defined in technical Appendix B.2. For general linear functions of f_t, the QKF provides a natural procedure to obtain the most probable vector of shocks. Let $\widetilde{\mathcal{B}} = (e_{v_2}, \mathcal{E}_v)$. The computable version of the IRF of Equation (A.15) is given by:

$$\mathcal{IRF}_{t,n}^{v_2 \to v_1} = e'_{v_1}\Psi^n \left[\mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1})\widetilde{\mathcal{B}}\left(\widetilde{\mathcal{B}}'\left[\mathrm{Vec}^{-1}(\Omega_0 + \Omega f_{t-1})\right]\widetilde{\mathcal{B}}\right)^{-1}\begin{pmatrix} s \\ 0 \end{pmatrix}\right]. \tag{A.17}$$

Again, the terms in the bracket are slightly modified such that $\mathrm{Vec}(XX')_{t|t} = \mathrm{Vec}(X_{t|t}X'_{t|t})$ and $z_{t|t} \geqslant 0$. To understand where Equation (A.17) comes from, consider the initial conditions f_{t-1} and the shocks are known without errors, so $P_{t-1,t-1} = 0$. Replacing the unknown quantities in Equation (A.15) by the values given by the QKF, the result is immediately obtained.

So far, we have only considered responses on variables that can be expressed as affine combinations of the extended vector of factors f_t. However, because of the closed-formedness of the conditional Laplace transform of f_t given its past, we can also compute the conditional expectation of any exponential-affine combination of f_t in closed-form. In general this requires the use of the multi-horizon conditional Laplace transform, which we detail in the Appendix A.5. In practice, we apply these formulas to obtain the responses of the ELB probabilities and the corresponding premia.

The average IRF can be computed in two different ways. First, we can apply Formula (A.15) to the initial condition $f_{t-1} = \left[\bar{X}', \left(\bar{X} \otimes \bar{X}\right)', \bar{z}\right]'$, where $\bar{X} := \mathbb{E}(X_t)$ and $\bar{z} = \mathbb{E}(z_t)$.[18] Second, we can simulate many initial conditions f_{t-1} using its marginal distribution, compute the IRFs using Formula (A.15) for each initial condition, and average over the responses. The two approaches are not equivalent since they flip the order of integration (see for example Gallant et al. (1993) or Koop et al. (1996)). We rely on the latter method in Section 4.5.

A.9 Parameter Estimates

18. It is worth mentioning that the initial condition $f_{t-1} = [\bar{X}', \text{Vec}(\bar{X}\bar{X}')', \mathbb{E}(z_t)]'$ is different from $f_{t-1} = \mathbb{E}(f_{t-1})$ since $\mathbb{E}[XX'] \neq \mathbb{E}(X)\mathbb{E}(X')$. However, once conditioning by $X_{t-1} = \bar{X}$, it follows directly that $\text{Vec}(X_{t-1}X_{t-1}') = \text{Vec}(\bar{X}\bar{X}')$ with probability one.

Table A.1: Parameter estimates: X_t dynamics

	estimates	std.		estimates	std.
μ_{π^*}	-0.0392*	(0.0237)	$\mu^{\mathbb{Q}}_{\pi^*}$	-0.0162	(0.0581)
μ_{σ}	0.0158	(0.0108)	$\mu^{\mathbb{Q}}_{\sigma}$	-0.0452**	(0.0186)
μ_{y_1}	0	–	$\mu^{\mathbb{Q}}_{y_1}$	-0.2521	(0.5486)
μ_{y_2}	0	–	$\mu^{\mathbb{Q}}_{y_2}$	0.1948*	(0.1056)
Φ_{π^*}	0.9014***	(0.0156)	$\Phi^{\mathbb{Q}}_{\pi^*}$	0.9639***	(0.006)
Φ_{σ,π^*}	0.0126***	(0.0049)	$\Phi^{\mathbb{Q}}_{\sigma,\pi^*}$	0.0011	(0.0017)
Φ_{y_1,π^*}	0	–	$\Phi^{\mathbb{Q}}_{y_1,\pi^*}$	-0.0361	(0.0459)
Φ_{y_2,π^*}	0	–	$\Phi^{\mathbb{Q}}_{y_2,\pi^*}$	-0.008	(0.0102)
$\Phi_{\pi^*,\sigma}$	0.0665***	(0.0232)	$\Phi^{\mathbb{Q}}_{\pi^*,\sigma}$	0.2155***	(0.0427)
Φ_{σ}	0.9732***	(0.0063)	$\Phi^{\mathbb{Q}}_{\sigma}$	0.9087***	(0.0195)
$\Phi_{y_1,\sigma}$	0	–	$\Phi^{\mathbb{Q}}_{y_1,\sigma}$	1.7502***	(0.254)
$\Phi_{y_2,\sigma}$	0	–	$\Phi^{\mathbb{Q}}_{y_2,\sigma}$	0.3881*	(0.2356)
Φ_{π^*,y_1}	-0.0016	(0.0012)	$\Phi^{\mathbb{Q}}_{\pi^*,y_1}$	-0.0171***	(0.0028)
Φ_{σ,y_1}	0	$(2{\cdot}10^{-4})$	$\Phi^{\mathbb{Q}}_{\sigma,y_1}$	0.0044***	(0.0012)
Φ_{y_1}	0.9826***	(0.0025)	$\Phi^{\mathbb{Q}}_{y_1}$	0.8364***	(0.0181)
Φ_{y_2,y_1}	0	–	$\Phi^{\mathbb{Q}}_{y_2,y_1}$	-0.0194	(0.0205)
Φ_{π^*,y_2}	0.0041***	(0.0012)	$\Phi^{\mathbb{Q}}_{\pi^*,y_2}$	-0.0031	(0.003)
Φ_{σ,y_2}	$-9{\cdot}10^{-4*}$	$(5{\cdot}10^{-4})$	$\Phi^{\mathbb{Q}}_{\sigma,y_2}$	0.003***	(0.001)
Φ_{y_1,y_2}	0.008**	(0.0033)	$\Phi^{\mathbb{Q}}_{y_1,y_2}$	-0.0339	(0.0255)
Φ_{y_2}	0.9974***	$(9{\cdot}10^{-4})$	$\Phi^{\mathbb{Q}}_{y_2}$	0.9885***	(0.0085)
Σ_{π^*}	0.1159***	(0.0136)	$\Sigma^{\mathbb{Q}}_{\pi^*}$ 0.1159***	(0.0136)	
Σ_{σ,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{\sigma,\pi^*}$	0	(0)
Σ_{y_1,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{y_1,\pi^*}$	0	(0)
Σ_{y_2,π^*}	0	–	$\Sigma^{\mathbb{Q}}_{y_2,\pi^*}$	0	(0)
Σ_{σ}	0.0151***	(0.0042)	$\Sigma^{\mathbb{Q}}_{\sigma}$	0.0151***	(0.0042)
$\Sigma_{y_1,\sigma}$	0	–	$\Sigma^{\mathbb{Q}}_{y_1,\sigma}$	0	(0)
$\Sigma_{y_2,\sigma}$	0	–	$\Sigma^{\mathbb{Q}}_{y_2,\sigma}$	0	(0)
Σ_{y_1}	1	–	$\Sigma^{\mathbb{Q}}_{y_1}$	1.0006***	$(2 \cdot 10^{-4})$
Σ_{y_2,y_1}	0	–	$\Sigma^{\mathbb{Q}}_{y_2,y_1}$	0	(0)
Σ_{y_2}	1	–	$\Sigma^{\mathbb{Q}}_{y_2}$	1	(0)

Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

Table A.2: Parameter estimates: short-rate and the prices of risk

	r_t dynamics *(parameters are divided by 1,200 except c, $\bar{\pi}$ and $\underline{r}$)*				
	estimates	std.		estimates	std.
α	0.4171***	(0.1692)	$\alpha^{\mathbb{Q}}$	0.4905**	(0.2081)
β_{π^*}	0.0025	(0.0058)	$\beta^{\mathbb{Q}}_{\pi^*}$	0.0027	(0.0062)
β_{σ}	-0.0576**	(0.0282)	$\beta^{\mathbb{Q}}_{\sigma}$	-0.0624**	(0.0305)
β_{y_1}	0.0398***	(0.0036)	$\beta^{\mathbb{Q}}_{y_1}$	0.0431***	(0.004)
β_{y_2}	0.0063	(0.0051)	$\beta^{\mathbb{Q}}_{y_2}$	0.0068	(0.0055)
κ	1.3327***	(0.2586)	$\kappa^{\mathbb{Q}}$	1.4453***	(0.2935)
ϕ	0.6421***	(0.0426)	$\phi^{\mathbb{Q}}$	0.7551***	(0.0455)
$c \cdot 1200$	1.0646***	(0.0697)	$c^{\mathbb{Q}} \cdot 1200$	1.2521***	(0.0771)
$\underline{r} \cdot 1200$	0.1313***	(0.0091)	$\bar{\pi} \cdot 100$	2.4336	–
	Prices of risk and measurement errors standard deviations				
	estimates	std.		estimates	std.
λ_{0,π^*}	0.1986	(0.4512)	λ_{0,y_1}	-0.2619	(0.5507)
$\lambda_{0,\sigma}$	-4.0353***	(1.4374)	λ_{0,y_2}	0.1932*	(0.1061)
λ_{1,π^*}	0.5400***	(0.1344)	λ_{1,π^*,y_1}	-0.134***	(0.031)
λ_{1,σ,π^*}	-0.7579**	(0.321)	λ_{1,σ,y_1}	0.2924***	(0.0612)
λ_{1,y_1,π^*}	-0.0361	(0.0459)	λ_{1,y_1}	-0.1467***	(0.0184)
λ_{1,y_2,π^*}	-0.008	(0.0102)	λ_{1,y_2,y_1}	-0.0194	(0.0206)
$\lambda_{1,\pi^*,\sigma}$	1.2857***	(0.4019)	λ_{1,π^*,y_2}	-0.062***	(0.025)
$\lambda_{1,\sigma}$	-4.2815***	(1.3907)	λ_{1,σ,y_2}	0.2543***	(0.0649)
$\lambda_{1,y_1,\sigma}$	1.7499***	(0.254)	λ_{1,y_1,y_2}	-0.0419*	(0.0242)
$\lambda_{1,y_2,\sigma}$	0.3881*	(0.2356)	λ_{1,y_2}	-0.0088	(0.0089)
λ_r	0.1406***	(0.0247)			
σ_R	0.0333***	$(6 \cdot 10^{-4})$	σ^*_R	0.0341***	$(6 \cdot 10^{-4})$
$\sigma^{(12)}_{\pi}$	0.4898	–	$\sigma^{(120)}_{\pi}$	0.3653	–
$\sigma^{(3)}_{S_R}$	0.2194	–	$\sigma^{(12)}_{S_R}$	0.4093	–
σ_{ZLB}	0.0457	–	$\sigma^{(120)}_{S_r}$	0.7085	–

Notes: Standard deviations are in parentheses and are calculated using the outer-product Hessian approximation. The '–' sign indicates that the parameter has been calibrated hence does not possess any standard deviation. Significance level: * <0.1, ** <0.05, *** <0.01.

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